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Dynamic solution of a multilayered orthotropic piezoelectric hollow cylinder for axisymmetric plane strain problems

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Abstract

The dynamic solution of a multilayered orthotropic piezoelectric infinite hollow cylinder in the state of axisymmetric plane strain is obtained. By the method of superposition, the solution is divided into two parts: one is quasi-static and the other is dynamic. The quasi-static part is derived by the state space method, and the dynamic part is obtained by the separation of variables method coupled with the initial parameter method as well as the orthogonal expansion technique. By using the obtained quasi-static and dynamic parts and the electric boundary conditions as well as the electric continuity conditions, a Volterra integral equation of the second kind with respect to a function of time is derived, which can be solved successfully by means of the interpolation method. The displacements, stresses and electric potentials are finally obtained. The present method is suitable for a multilayered orthotropic piezoelectric infinite hollow cylinder consisting of arbitrary layers and subjected to arbitrary axisymmetric dynamic loads. Numerical results are finally presented and discussed.

Keywords: Axisymmetric; Elastodynamics; Multilayered hollow cylinder; Orthotropy; Piezoelectric

1. Introduction

The piezoelectric materials have been used widely in modern intelligent structural systems due to their special electro-mechanical coupling effect (Rao and Sunar, 1994). For the homogeneous (single layer)

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piezoelectric media, Paul (1966) obtained the frequency equation of a piezoelectric cylindrical shell. The free vibrations of the radially and axially polarized piezoelectric ceramic cylinders have been studied by Adelman et al. (1975a,b). The electroelastic waves in solid and hollow cylinders have also been studied (Paul and Raju, 1982; Shul'ga et al., 1984; Paul and Venkatesan, 1987). Ding et al. (1997a,b) further investigated the three-dimensional free vibrations of empty and compressible fluid filled piezoelectric cylindrical shells, respectively. Lin (1998) analyzed the coupled vibration of a piezoelectric ceramic disk resonator. Ding et al. (2003a) obtained the dynamic solution of a piezoelectric hollow cylinder for axisymmetric plane strain problems. There are also a number of investigations on multilayered piezoelectric media. Kharouf and Heyliger (1994) and Heyliger and Ramirez (2000) dealt with the free vibration problems of laminated piezoelectric cylinders and discs, respectively. Chen (2000) considered the free vibration of (multilayered) non-homogeneous piezoceramic hollow spheres by employing a separation formulation for displacements. Chen (2001) developed a state-space method for free vibration analysis of laminated piezoelectric hollow spheres. Li et al. (2001) studied the free vibration of a piezoelectric laminated cylindrical shell under hydrostatic pressure. The axisymmetric waves in layered piezoelectric rods have also been studied by Nayfeh et al. (2000). Recently, the transient plane strain response of a multilayered isotropic elastic hollow cylinder has been successfully solved by Yin and Yue (2002). Heyliger (1997) obtained the three-dimensional solution for the static problem of a finite laminated piezoelectric cylinder with its ends simply supported. Siao et al. (1994) investigated the frequency spectra of laminated piezoelectric cylinders. The transient responses, a dynamic problem differing from the frequency spectra analysis, for a multilayered orthotropic piezoelectric infinite hollow cylinder has not been reported.

In this paper, the analysis of homogeneous (single layered) piezoelectric hollow cylinders (Ding et al., 2003a) is extended to solve the dynamic responses of the multilayered ones. The dynamic solution of a multilayered orthotropic piezoelectric hollow cylinder in the state of axisymmetric plane strain is first divided into two parts by the method of superposition: one is quasi-static and the other is dynamic. Then by the state space method coupled with the initial parameter method, the static part and the dynamic part can be obtained via operating the matrix of order two only. The present method provides an efficient way to solve the dynamic problem for multilayered piezoelectric hollow cylinder because the static part is obtained in an explicit form and the resulting eigenequation is very simple, which greatly facilitates solving for eigenroots quickly. By using the electric boundary conditions and electric continuity conditions, a Volterra integral equation of the second kind is derived, which can be solved successfully by the interpolation method developed by Ding et al. (2003b) recently.

2. Basic equations and their non-dimensional forms

Consider an infinite piezoelectric hollow cylinder composed of n layers with inner radius $r_0 = a$ and outer radius $r_n = b$, as shown in Fig. 1. The first layer is the innermost and the nth layer is the outermost. The inner and outer radii of the ith layer are denoted as r_{i-1} and r_i, respectively.

We formulate the problem in the cylindrical coordinate system (r, θ, z). For the axisymmetric plane strain case, we have $u_\theta^{(i)} = u_z^{(i)} = 0$, $u_r^{(i)} = u_r^{(i)}(r,t)$ for the components of displacement, and $\Phi^{(i)} = \Phi^{(i)}(r,t)$ for the component electric potential in the ith layer ($r_{i-1} \leqslant r \leqslant r_i$). The non-zero components of strain are

$$\gamma_{rr}^{(i)} = \frac{\partial u_r^{(i)}}{\partial r}, \qquad \gamma_{\theta\theta}^{(i)} = \frac{u_r^{(i)}}{r}. \tag{1}$$

If each layer characterizes material orthotropy, then the constitutive relations of the ith layer are (Adelman et al., 1975a)

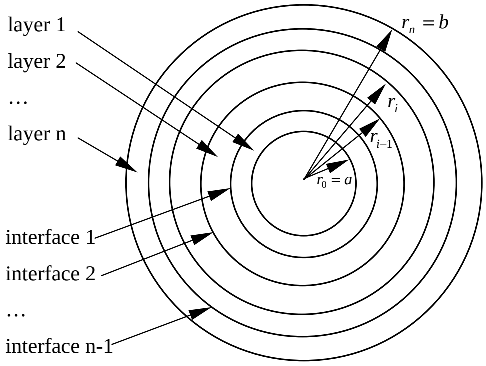

Fig. 1. Geometry of a multilayered hollow cylinder.

$$\begin{aligned}
\sigma_{\theta\theta}^{(i)} &= c_{11}^{(i)}\frac{u_r^{(i)}}{r} + c_{13}^{(i)}\frac{\partial u_r^{(i)}}{\partial r} + e_{31}^{(i)}\frac{\partial \Phi^{(i)}}{\partial r},\\
\sigma_{zz}^{(i)} &= c_{12}^{(i)}\frac{u_r^{(i)}}{r} + c_{23}^{(i)}\frac{\partial u_r^{(i)}}{\partial r} + e_{32}^{(i)}\frac{\partial \Phi^{(i)}}{\partial r},\\
\sigma_{rr}^{(i)} &= c_{13}^{(i)}\frac{u_r^{(i)}}{r} + c_{33}^{(i)}\frac{\partial u_r^{(i)}}{\partial r} + e_{33}^{(i)}\frac{\partial \Phi^{(i)}}{\partial r},\\
D_{rr}^{(i)} &= e_{31}^{(i)}\frac{u_r^{(i)}}{r} + e_{33}^{(i)}\frac{\partial u_r^{(i)}}{\partial r} - \varepsilon_{33}^{(i)}\frac{\partial \Phi^{(i)}}{\partial r},
\end{aligned} \tag{2}$$

where $\sigma_{jj}^{(i)}$ $(j = r, \theta, z)$ and $D_{rr}^{(i)}$ are the components of stress and radial electric displacement, $c_{jm}^{(i)}$ $(j, m = 1, 2, 3)$, $e_{3j}^{(i)}$ $(j = 1, 2, 3)$ and $\varepsilon_{33}^{(i)}$ are the elastic, piezoelectric and dielectric constants of the ith layer, respectively. The equation of motion of the ith layer is

$$\frac{\partial \sigma_{rr}^{(i)}}{\partial r} + \frac{\sigma_{rr}^{(i)} - \sigma_{\theta\theta}^{(i)}}{r} = \rho^{(i)}\frac{\partial^2 u_r^{(i)}}{\partial t^2}, \tag{3}$$

where $\rho^{(i)}$ is the mass density of the ith layer. In the absence of free charge density, the charge equation of electrostatics is

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rD_{rr}^{(i)}\right] = 0. \tag{4}$$

The boundary conditions are

$$\sigma_{rr}^{(1)}(a, t) = q_0(t), \qquad \sigma_{rr}^{(n)}(b, t) = q_n(t), \tag{5a}$$

$$\Phi^{(1)}(a, t) = \Phi_0(t), \qquad \Phi^{(n)}(b, t) = \Phi_n(t), \tag{5b}$$

where $q_0(t)$ and $q_n(t)$ are prescribed dynamic pressures acting on the inner and outer surfaces, respectively. And $\Phi_0(t)$ and $\Phi_n(t)$ are known electric potentials applied on the inner and outer surfaces, respectively. The continuity conditions at the interfaces can be expressed as

$$\sigma_{rr}^{(i+1)}(r_i, t) = \sigma_{rr}^{(i)}(r_i, t), \quad u_r^{(i+1)}(r_i, t) = u_r^{(i)}(r_i, t) \qquad (i = 1, 2, \ldots, n-1); \tag{6a}$$

$$\Phi^{(i+1)}(r_i, t) = \Phi^{(i)}(r_i, t), \quad D_{rr}^{(i+1)}(r_i, t) = D_{rr}^{(i)}(r_i, t) \qquad (i = 1, 2, \ldots, n-1). \tag{6b}$$

The initial conditions $(t = 0)$ are

$$u_r^{(i)}(r, 0) = U_0^{(i)}(r), \quad \dot{u}_r^{(i)}(r, 0) = V_0^{(i)}(r) \qquad (i = 1, 2, \ldots, n), \tag{7}$$

where $U_0^{(i)}(r)$ and $V_0^{(i)}(r)$ are known functions of the radial coordinate r and a dot over a quantity denotes its partial derivative with respect to time t.

For convenience, the following non-dimensional quantities are introduced,

$$\begin{gathered}
c_{11P}^{(i)} = \frac{c_{11}^{(i)}}{c_{33}^{(1)}}, \quad c_{12P}^{(i)} = \frac{c_{12}^{(i)}}{c_{33}^{(1)}}, \quad c_{13P}^{(i)} = \frac{c_{13}^{(i)}}{c_{33}^{(1)}}, \quad c_{23P}^{(i)} = \frac{c_{23}^{(i)}}{c_{33}^{(1)}}, \quad c_{33P}^{(i)} = \frac{c_{33}^{(i)}}{c_{33}^{(1)}}, \\
e_1^{(i)} = \frac{e_{31}^{(i)}}{\sqrt{c_{33}^{(1)}\varepsilon_{33}^{(1)}}}, \quad e_2^{(i)} = \frac{e_{32}^{(i)}}{\sqrt{c_{33}^{(1)}\varepsilon_{33}^{(1)}}}, \quad e_3^{(i)} = \frac{e_{33}^{i}}{\sqrt{c_{33}^{(1)}\varepsilon_{33}^{(1)}}}, \quad \varepsilon_3^{(i)} = \frac{\varepsilon_{33}^{(i)}}{\varepsilon_{33}^{(1)}}, \\
\bar{\rho}^{(i)} = \frac{\rho^{(i)}}{\rho^{(1)}}, \quad u^{(i)} = \frac{u_r^{(i)}}{b}, \quad \sigma_j^{(i)} = \frac{\sigma_{jj}^{(i)}}{c_{33}^{(1)}} \quad (j = r, \theta, z), \quad D_r^{(i)} = \frac{D_{rr}^{(i)}}{\sqrt{c_{33}^{(1)}\varepsilon_{33}^{(1)}}}, \\
\phi^{(i)} = \sqrt{\frac{\varepsilon_{33}^{(1)}}{c_{33}^{(1)}}}\frac{\Phi^{(i)}}{b}, \quad \phi_0 = \sqrt{\frac{\varepsilon_{33}^{(1)}}{c_{33}^{(1)}}}\frac{\Phi_0}{b}, \quad \phi_n = \sqrt{\frac{\varepsilon_{33}^{(1)}}{c_{33}^{(1)}}}\frac{\Phi_n}{b}, \\
u_0^{(i)} = \frac{U_0^{(i)}}{b}, \quad v_0^{(i)} = \frac{V_0^{(i)}}{c_v}, \quad p_0 = \frac{q_0}{c_{33}^{(1)}}, \quad p_n = \frac{q_n}{c_{33}^{(1)}}, \\
\xi = \frac{r}{b}, \quad \xi_i = \frac{r_i}{b} \quad (i = 0, 1, \ldots, n), \quad c_v = \sqrt{\frac{c_{33}^{(1)}}{\rho^{(1)}}}, \quad \tau = \frac{c_v}{b}t.
\end{gathered} \tag{8}$$

By virtue of Eq. (8), Eqs. (2)–(4) can be rewritten as

$$\begin{aligned}
\sigma_\theta^{(i)} &= c_{11P}^{(i)}\frac{u^{(i)}}{\xi} + c_{13P}^{(i)}\frac{\partial u^{(i)}}{\partial \xi} + e_1^{(i)}\frac{\partial \phi^{(i)}}{\partial \xi}, \\
\sigma_z^{(i)} &= c_{12P}^{(i)}\frac{u^{(i)}}{\xi} + c_{23P}^{(i)}\frac{\partial u^{(i)}}{\partial \xi} + e_2^{(i)}\frac{\partial \phi^{(i)}}{\partial \xi}, \\
\sigma_r^{(i)} &= c_{13P}^{(i)}\frac{u^{(i)}}{\xi} + c_{33P}^{(i)}\frac{\partial u^{(i)}}{\partial \xi} + e_3^{(i)}\frac{\partial \phi^{(i)}}{\partial \xi}, \\
D_r^{(i)} &= e_1^{(i)}\frac{u^{(i)}}{\xi} + e_3^{(i)}\frac{\partial u^{(i)}}{\partial \xi} - \varepsilon_3^{(i)}\frac{\partial \phi^{(i)}}{\partial \xi},
\end{aligned} \tag{9}$$

$$\frac{\partial \sigma_r^{(i)}}{\partial \xi} + \frac{\sigma_r^{(i)} - \sigma_\theta^{(i)}}{\xi} = \bar{\rho}^{(i)}\frac{\partial^2 u^{(i)}}{\partial \tau^2}, \tag{10}$$

$$\frac{1}{\xi}\frac{\partial}{\partial \xi}\left[\xi D_r^{(i)}\right] = 0, \tag{11}$$

and the boundary conditions (5a) and (5b), the continuity conditions (6a) and (6b) as well as the initial condition (7) can be rewritten as

$$\sigma_r^{(1)}(\xi_0, \tau) = p_0(\tau), \qquad \sigma_r^{(n)}(\xi_n, \tau) = p_n(\tau), \tag{12a}$$

$$\phi^{(1)}(\xi_0, \tau) = \phi_0(\tau), \qquad \phi^{(n)}(\xi_n, \tau) = \phi_n(\tau), \tag{12b}$$

$$\sigma_r^{(i+1)}(\xi_i,\tau) = \sigma_r^{(i)}(\xi_i,\tau), \quad u^{(i+1)}(\xi_i,\tau) = u^{(i)}(\xi_i,\tau) \qquad (i = 1,2,\ldots,n-1), \tag{13a}$$

$$\phi^{(i+1)}(\xi_i,\tau) = \phi^{(i)}(\xi_i,\tau), \quad D_r^{(i+1)}(\xi_i,\tau) = D_r^{(i)}(\xi_i,\tau) \qquad (i = 1,2,\ldots,n-1), \tag{13b}$$

$$u^{(i)}(\xi,0) = u_0^{(i)}(\xi), \quad \dot{u}^{(i)}(\xi,0) = v_0^{(i)}(\xi) \qquad (i = 1,2,\ldots,n). \tag{14}$$

In Eq. (14) and hereafter, a dot over a quantity denotes its partial derivative with respect to the non-dimensional time τ.

3. Solution technique by the method of superposition

First, the first three equations in Eqs. (9) and (10) can be rewritten as

$$\begin{aligned}
\Sigma_\theta^{(i)} &= c_{11P}^{(i)}u^{(i)} + c_{13P}^{(i)}\nabla u^{(i)} + e_1^{(i)}\nabla\phi^{(i)},\\
\Sigma_z^{(i)} &= c_{12P}^{(i)}u^{(i)} + c_{23P}^{(i)}\nabla u^{(i)} + e_2^{(i)}\nabla\phi^{(i)},\\
\Sigma_r^{(i)} &= c_{13P}^{(i)}u^{(i)} + c_{33P}^{(i)}\nabla u^{(i)} + e_3^{(i)}\nabla\phi^{(i)},
\end{aligned} \tag{15}$$

$$\nabla\Sigma_r^{(i)} - \Sigma_\theta^{(i)} = \bar{\rho}^{(i)}\xi^2\frac{\partial^2 u^{(i)}}{\partial\tau^2}, \tag{16}$$

where

$$\Sigma_r^{(i)} = \xi\sigma_r^{(i)}, \qquad \Sigma_\theta^{(i)} = \xi\sigma_\theta^{(i)}, \qquad \Sigma_z^{(i)} = \xi\sigma_z^{(i)}, \qquad \nabla = \xi\frac{\partial}{\partial\xi}. \tag{17}$$

By means of Eq. (17), Eqs. (12a) and (13a) can be rewritten as

$$\Sigma_r^{(1)}(\xi_0,\tau) = \xi_0 p_0(\tau), \qquad \Sigma_r^{(n)}(\xi_n,\tau) = \xi_n p_n(\tau), \tag{18}$$

$$\Sigma_r^{(i+1)}(\xi_i,\tau) = \Sigma_r^{(i)}(\xi_i,\tau), \quad u^{(i+1)}(\xi_i,\tau) = u^{(i)}(\xi_i,\tau) \qquad (i = 1,2,\ldots,n-1). \tag{19}$$

From Eq. (11), we have

$$D_r^{(i)}(\xi,\tau) = \frac{1}{\xi}\eta^{(i)}(\tau) \quad (i = 1,2,\ldots,n), \tag{20}$$

where $\eta^{(i)}(\tau)$ is an unknown function with respect to the non-dimensional time τ. From Eq. (20) and the second equation in Eq. (13b), we have

$$\eta^{(1)}(\tau) = \eta^{(2)}(\tau) = \cdots = \eta^{(n)}(\tau) = \eta(\tau). \tag{21}$$

By utilizing Eqs. (20) and (21), we can obtain the following equation from the fourth equation in Eq. (9):

$$\nabla\phi^{(i)} = \frac{e_1^{(i)}}{\varepsilon_3^{(i)}}u^{(i)} + \frac{e_3^{(i)}}{\varepsilon_3^{(i)}}\nabla u^{(i)} - \frac{1}{\varepsilon_3^{(i)}}\frac{\eta(\tau)}{\xi}. \tag{22}$$

Substituting Eq. (22) into Eq. (15) yields

$$\begin{aligned}
\Sigma_\theta^{(i)} &= c_{11D}^{(i)}u^{(i)} + c_{13D}^{(i)}\nabla u^{(i)} - e_{1D}^{(i)}\eta(\tau),\\
\Sigma_z^{(i)} &= c_{12D}^{(i)}u^{(i)} + c_{23D}^{(i)}\nabla u^{(i)} - e_{2D}^{(i)}\eta(\tau),\\
\Sigma_r^{(i)} &= c_{13D}^{(i)}u^{(i)} + c_{33D}^{(i)}\nabla u^{(i)} - e_{3D}^{(i)}\eta(\tau),
\end{aligned} \tag{23}$$

where

$$c_{11D}^{(i)}=c_{11P}^{(i)}+\frac{e_1^{(i)}e_1^{(i)}}{\varepsilon_3^{(i)}},\qquad c_{12D}^{(i)}=c_{12P}^{(i)}+\frac{e_1^{(i)}e_2^{(i)}}{\varepsilon_3^{(i)}},\qquad c_{13D}^{(i)}=c_{13P}^{(i)}+\frac{e_1^{(i)}e_3^{(i)}}{\varepsilon_3^{(i)}},$$
$$c_{23D}^{(i)}=c_{23P}^{(i)}+\frac{e_2^{(i)}e_3^{(i)}}{\varepsilon_3^{(i)}},\qquad c_{33D}^{(i)}=c_{33P}^{(i)}+\frac{e_3^{(i)}e_3^{(i)}}{\varepsilon_3^{(i)}},\tag{24}$$
$$e_{1D}^{(i)}=\frac{e_1^{(i)}}{\varepsilon_3^{(i)}},\qquad e_{2D}^{(i)}=\frac{e_2^{(i)}}{\varepsilon_3^{(i)}},\qquad e_{3D}^{(i)}=\frac{e_3^{(i)}}{\varepsilon_3^{(i)}}.$$

Then, according to the method of superposition (Berry and Naghdi, 1956), the displacement and stresses can be assumed as

$$u^{(i)}=u_s^{(i)}+u_d^{(i)},\qquad \Sigma_r^{(i)}=\Sigma_{rs}^{(i)}+\Sigma_{rd}^{(i)},\qquad \Sigma_\theta^{(i)}=\Sigma_{\theta s}^{(i)}+\Sigma_{\theta d}^{(i)},\tag{25}$$

where $u_s^{(i)}$, $\Sigma_{rs}^{(i)}$ and $\Sigma_{\theta s}^{(i)}$ are the quasi-static solutions satisfying the following equations:

$$\Sigma_{\theta s}^{(i)}=c_{11D}^{(i)}u_s^{(i)}+c_{13D}^{(i)}\nabla u_s^{(i)}-e_{1D}^{(i)}\eta(\tau),\qquad \Sigma_{rs}^{(i)}=c_{13D}^{(i)}u_s^{(i)}+c_{33D}^{(i)}\nabla u_s^{(i)}-e_{3D}^{(i)}\eta(\tau),\tag{26}$$

$$\nabla\Sigma_{rs}^{(i)}-\Sigma_{\theta s}^{(i)}=0,\tag{27}$$

$$\Sigma_{rs}^{(1)}(\xi_0,\tau)=\xi_0 p_0(\tau),\qquad \Sigma_{rs}^{(n)}(\xi_n,\tau)=\xi_n p_n(\tau),\tag{28}$$

$$\Sigma_{rs}^{(i+1)}(\xi_i,\tau)=\Sigma_{rs}^{(i)}(\xi_i,\tau),\quad u_s^{(i+1)}(\xi_i,\tau)=u_s^{(i)}(\xi_i,\tau)\qquad (i=1,2,\ldots,n-1).\tag{29}$$

Substituting Eq. (25) into the first and third equations in Eq. (23), Eqs. (16), (18), (19) and (14), and utilizing Eqs. (26)–(29), yields

$$\Sigma_{\theta d}^{(i)}=c_{11D}^{(i)}u_d^{(i)}+c_{13D}^{(i)}\nabla u_d^{(i)},\qquad \Sigma_{rd}^{(i)}=c_{13D}^{(i)}u_d^{(i)}+c_{33D}^{(i)}\nabla u_d^{(i)},\tag{30}$$

$$\nabla\Sigma_{rd}^{(i)}-\Sigma_{\theta d}^{(i)}=\bar\rho^{(i)}\xi^2\left[\ddot u_d^{(i)}+\ddot u_s^{(i)}\right],\tag{31}$$

$$\Sigma_{rd}^{(1)}(\xi_0,\tau)=0,\qquad \Sigma_{rd}^{(n)}(\xi_n,\tau)=0,\tag{32}$$

$$\Sigma_{rd}^{(i+1)}(\xi_i,\tau)=\Sigma_{rd}^{(i)}(\xi_i,\tau),\quad u_d^{(i+1)}(\xi_i,\tau)=u_d^{(i)}(\xi_i,\tau)\qquad (i=1,2,\ldots,n-1),\tag{33}$$

$$u_d^{(i)}(\xi,0)=u_0^{(i)}(\xi)-u_s^{(i)}(\xi,0),\quad \dot u_d^{(i)}(\xi,0)=v_0^{(i)}(\xi)-\dot u_s^{(i)}(\xi,0)\qquad (i=1,2,\ldots,n).\tag{34}$$

4. Quasi-static solution

The second equation in Eq. (26) can be rewritten as

$$\nabla u_s^{(i)}=a_1^{(i)}u_s^{(i)}+a_2^{(i)}\Sigma_{rs}^{(i)}+a_5^{(i)}\eta(\tau),\tag{35}$$

where

$$a_1^{(i)}=-\frac{c_{13D}^{(i)}}{c_{33D}^{(i)}},\qquad a_2^{(i)}=\frac{1}{c_{33D}^{(i)}},\qquad a_5^{(i)}=\frac{e_{3D}^{(i)}}{c_{33D}^{(i)}}.\tag{36}$$

Substituting the first equation in Eq. (26) into Eq. (27) and utilizing Eq. (35), we obtain

$$\nabla\Sigma_{rs}^{(i)} = a_3^{(i)} u_s^{(i)} + a_4^{(i)}\Sigma_{rs}^{(i)} + a_6^{(i)}\eta(\tau), \tag{37}$$

where

$$a_3^{(i)} = c_{11D}^{(i)} + c_{13D}^{(i)} a_1^{(i)}, \qquad a_4^{(i)} = c_{13D}^{(i)} a_2^{(i)}, \qquad a_6^{(i)} = c_{13D}^{(i)} a_5^{(i)} - e_{1D}^{(i)}. \tag{38}$$

Eqs. (35) and (37) can be rewritten in a matrix form as

$$\nabla\{\mathbf{X}^{(i)}(\xi,\tau)\} = [N^{(i)}]\{\mathbf{X}^{(i)}(\xi,\tau)\} + \{L^{(i)}\}\eta(\tau), \tag{39}$$

where

$$\{\mathbf{X}^{(i)}(\xi,\tau)\} = \begin{Bmatrix} u_s^{(i)}(\xi,\tau) \\ \Sigma_{rs}^{(i)}(\xi,\tau) \end{Bmatrix}, \qquad [N^{(i)}] = \begin{bmatrix} a_1^{(i)} & a_2^{(i)} \\ a_3^{(i)} & a_4^{(i)} \end{bmatrix}, \qquad \{L^{(i)}\} = \begin{Bmatrix} a_5^{(i)} \\ a_6^{(i)} \end{Bmatrix}. \tag{40}$$

The solution of Eq. (39) is

$$\{\mathbf{X}^{(i)}(\xi,\tau)\} = [T^{(i)}(\xi)]\left[\{\mathbf{X}^{(i)}(\xi_{i-1},\tau)\} + \{G^{(i)}(\xi)\}\eta(\tau)\right], \tag{41}$$

where

$$[T^{(i)}(\xi)] = \left(\frac{\xi}{\xi_{i-1}}\right)^{[N^{(i)}]}, \qquad \{G^{(i)}(\xi)\} = \int_{\xi_{i-1}}^{\xi} [T^{(i)}(\zeta)]^{-1}\{L^{(i)}\}\frac{1}{\zeta}\,\mathrm{d}\zeta, \tag{42}$$

in which $[T^{(i)}(\xi)]$ is a 2×2 matrix, and $\{G^{(i)}(\xi)\}$ is a column vector with two elements $G_1^{(i)}(\xi,\tau)$ and $G_2^{(i)}(\xi,\tau)$. In view of the first equation in Eq. (40), the continuity conditions in Eq. (29) can be rewritten as

$$\{\mathbf{X}^{(i+1)}(\xi_i,\tau)\} = \{\mathbf{X}^{(i)}(\xi_i,\tau)\} \quad (i = 1,2,\ldots,n-1). \tag{43}$$

Setting $\xi = \xi_i$ in Eq. (41) and repeatedly using (43), we can derive the following equation:

$$\{\mathbf{X}^{(i)}(\xi_i,\tau)\} = [H^{(i)}]\{\mathbf{X}^{(1)}(\xi_0,\tau)\} + \{M^{(i)}\}\eta(\tau) \quad (i = 1,2,\ldots,n), \tag{44}$$

where

$$[H^{(i)}] = \left[\widehat{T}_1^{(i)}\right], \quad \{M^{(i)}\} = \sum_{m=1}^{i}\left[\widehat{T}_m^{(i)}\right]\{G^{(m)}(\xi_m)\}, \quad \left[\widehat{T}_m^{(i)}\right] = \prod_{j=i}^{m}[T^{(j)}(\xi_j)] \qquad (m = 1,2,\ldots,i), \tag{45}$$

in which $[H^{(i)}]$ is a 2×2 matrix and $\{M^{(i)}\}$ is a column vector with two elements $M_1^{(i)}(\tau)$ and $M_2^{(i)}(\tau)$. Setting $i = n$ in Eq. (44) and noticing Eq. (28), gives

$$\begin{Bmatrix} u_s^{(n)}(\xi_n,\tau) \\ \xi_n p_n(\tau) \end{Bmatrix} = \begin{bmatrix} H_{11}^{(n)} & H_{12}^{(n)} \\ H_{21}^{(n)} & H_{22}^{(n)} \end{bmatrix}\begin{Bmatrix} u_s^{(1)}(\xi_0,\tau) \\ \xi_0 p_0(\tau) \end{Bmatrix} + \begin{Bmatrix} M_1^{(n)} \\ M_2^{(n)} \end{Bmatrix}\eta(\tau). \tag{46}$$

From the second equation in Eq. (46), we have

$$u_s^{(1)}(\xi_0,\tau) = \left[\xi_n p_n(\tau) - H_{22}^{(n)}\xi_0 p_0(\tau) - M_2^{(n)}\eta(\tau)\right]\Big/ H_{21}^{(n)}. \tag{47}$$

Thus, Eq. (41) can be rewritten as

$$\begin{Bmatrix} u_s^{(i)}(\xi,\tau) \\ \Sigma_{rs}^{(i)}(\xi,\tau) \end{Bmatrix} = \begin{bmatrix} T_{11}^{(i)}(\xi) & T_{12}^{(i)}(\xi) \\ T_{21}^{(i)}(\xi) & T_{22}^{(i)}(\xi) \end{bmatrix}\left(\begin{bmatrix} H_{11}^{(i-1)} & H_{12}^{(i-1)} \\ H_{21}^{(i-1)} & H_{22}^{(i-1)} \end{bmatrix}\begin{Bmatrix} u_s^{(1)}(\xi_0,\tau) \\ \xi_0 p_0(\tau) \end{Bmatrix} + \begin{Bmatrix} M_1^{(i-1)} \\ M_2^{(i-1)} \end{Bmatrix}\eta(\tau) + \begin{Bmatrix} G_1^{(i)}(\xi) \\ G_2^{(i)}(\xi) \end{Bmatrix}\eta(\tau)\right). \tag{48}$$

Then the following equation from the first equation in Eq. (48) can be derived by utilizing Eq. (47),

$$u_s^{(i)}(\xi,\tau) = f_1^{(i)}(\xi)p_0(\tau) + f_2^{(i)}(\xi)p_n(\tau) + f_3^{(i)}(\xi)\eta(\tau), \tag{49}$$

where

$$\begin{aligned}
f_1^{(i)}(\xi) &= \xi_0\left\{\left[H_{12}^{(i-1)} - \frac{H_{22}^{(n)}}{H_{21}^{(n)}}H_{11}^{(i-1)}\right]T_{11}^{(i)}(\xi) + \left[H_{22}^{(i-1)} - \frac{H_{22}^{(n)}}{H_{21}^{(n)}}H_{21}^{(i-1)}\right]T_{12}^{(i)}(\xi)\right\},\\
f_2^{(i)}(\xi) &= \frac{\xi_n}{H_{21}^{(n)}}\left[H_{11}^{(i-1)}T_{11}^{(i)}(\xi) + H_{21}^{(i-1)}T_{12}^{(i)}(\xi)\right],\\
f_3^{(i)}(\xi) &= T_{11}^{(i)}(\xi)\left[M_1^{(i-1)} - \frac{M_2^{(n)}}{H_{21}^{(n)}}H_{11}^{(i-1)} + G_1^{(i)}(\xi)\right] + T_{12}^{(i)}(\xi)\left[M_2^{(i-1)} - \frac{M_2^{(n)}}{H_{21}^{(n)}}H_{21}^{(i-1)} + G_2^{(i)}(\xi)\right].
\end{aligned}\tag{50}$$

In the following, we will give the expressions of the elements in matrix $[T^{(i)}(\xi)]$. According to Cayley–Hamilton theorem (Deif, 1982), the first equation in Eq. (42) can be expressed as

$$[T^{(i)}(\xi)] = (\xi/\xi_{i-1})^{[N^{(i)}]} = E_0^{(i)}(\xi)\mathbf{I} + E_1^{(i)}(\xi)[N^{(i)}], \tag{51}$$

where $\mathbf{I}$ denotes a 2×2 unit matrix and $E_0^{(i)}(\xi)$ and $E_1^{(i)}(\xi)$ are determined by

$$(\xi/\xi_{i-1})^{\lambda_1^{(i)}} = E_0^{(i)}(\xi) + \lambda_1^{(i)}E_1^{(i)}(\xi), \qquad (\xi/\xi_{i-1})^{\lambda_2^{(i)}} = E_0^{(i)}(\xi) + \lambda_2^{(i)}E_1^{(i)}(\xi), \tag{52}$$

in which $\lambda_1^{(i)}$ and $\lambda_2^{(i)}$ are two eigenvalues of $[N^{(i)}]$, i.e.

$$\begin{aligned}
\lambda_1^{(i)} &= \frac{a_1^{(i)} + a_4^{(i)}}{2} - \frac{\sqrt{\left[a_1^{(i)} - a_4^{(i)}\right]^2 + 4a_2^{(i)}a_3^{(i)}}}{2},\\
\lambda_2^{(i)} &= \frac{a_1^{(i)} + a_4^{(i)}}{2} + \frac{\sqrt{\left[a_1^{(i)} - a_4^{(i)}\right]^2 + 4a_2^{(i)}a_3^{(i)}}}{2}.
\end{aligned}\tag{53}$$

Thus from Eq. (52), we obtain

$$\begin{aligned}
E_0^{(i)}(\xi) &= \frac{1}{\lambda_2^{(i)} - \lambda_1^{(i)}}\left[\lambda_2^{(i)}(\xi/\xi_{i-1})^{\lambda_1^{(i)}} - \lambda_1^{(i)}(\xi/\xi_{i-1})^{\lambda_2^{(i)}}\right],\\
E_1^{(i)}(\xi) &= \frac{1}{\lambda_2^{(i)} - \lambda_1^{(i)}}\left[(\xi/\xi_{i-1})^{\lambda_2^{(i)}} - (\xi/\xi_{i-1})^{\lambda_1^{(i)}}\right].
\end{aligned}\tag{54}$$

5. Dynamic solution

Substituting Eq. (30) into Eq. (31) and utilizing Eq. (49), we have

$$\frac{\partial^2 u_d^{(i)}}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial u_d^{(i)}}{\partial\xi} - \frac{\mu_i^2}{\xi^2}u_d^{(i)} = \frac{1}{c_i^2}\left[\frac{\partial^2 u_d^{(i)}}{\partial\tau^2} + f_1^{(i)}(\xi)\frac{\mathrm{d}^2 p_0(\tau)}{\mathrm{d}\tau^2} + f_2^{(i)}(\xi)\frac{\mathrm{d}^2 p_n(\tau)}{\mathrm{d}\tau^2} + f_3^{(i)}(\xi)\frac{\mathrm{d}^2\eta(\tau)}{\mathrm{d}\tau^2}\right], \tag{55}$$

where

$$\mu_i = \sqrt{\frac{c_{11D}^{(i)}}{c_{33D}^{(i)}}}, \qquad c_i = \sqrt{\frac{c_{33D}^{(i)}}{\bar{\rho}^{(i)}}}. \tag{56}$$

By means of separation of variables method, $u_d^{(i)}(\xi, \tau)$ can be assumed as

$$u_d^{(i)}(\xi, \tau) = \sum_{m=1}^{\infty} R_m^{(i)}(\xi)\Omega_m(\tau), \tag{57}$$

where $\Omega_m(\tau)$ is an undetermined function. According to the orthogonal expansion technique and noticing the differential form at the left-hand side of Eq. (55), we know that $R_m^{(i)}(\xi)$ must be a linear combination of $J_{\mu_i}(k_m^i\xi)$ and $Y_{\mu_i}(k_m^i\xi)$. Here $J_{\mu_i}(\cdot)$ and $Y_{\mu_i}(\cdot)$ are Bessel functions of the first and second kinds of order μ_i, and

$$k_m^i = \frac{\omega_m}{c_i}, \tag{58}$$

where ω_m is a series of undetermined positive real numbers.

Substituting Eq. (57) into the second equation in Eq. (30) yields

$$\Sigma_{rd}^{(i)}(\xi, \tau) = \sum_{m=1}^{\infty} \sigma_m^{(i)}(\xi)\Omega_m(\tau), \tag{59}$$

where

$$\sigma_m^{(i)}(\xi) = c_{33D}^{(i)}\nabla R_m^{(i)}(\xi) + c_{13D}^{(i)} R_m^{(i)}(\xi). \tag{60}$$

Substituting Eqs. (57) and (59) into Eqs. (32) and (33) gives

$$\sigma_m^{(1)}(\xi_0) = 0, \quad \sigma_m^{(n)}(\xi_n) = 0 \qquad (m = 1, 2, \ldots, \infty), \tag{61}$$

$$\sigma_m^{(i+1)}(\xi_i) = \sigma_m^{(i)}(\xi_i), \quad R_m^{(i+1)}(\xi_i) = R_m^{(i)}(\xi_i) \qquad (m = 1, 2, \ldots, \infty; \; i = 1, 2, \ldots, n). \tag{62}$$

The above equation indicates that using the initial parameter method to construct $R_m^{(i)}(\xi)$ and $\sigma_m^{(i)}(\xi)$ will be beneficial to the introduction of the boundary conditions and the realization of the continuity conditions at the interfaces. To do so, the following two principles should be applied: (1) $R_m^{(i)}(\xi)$ must be the linear combination of $J_{\mu_i}(k_m^i\xi)$ and $Y_{\mu_i}(k_m^i\xi)$; and (2) Eq. (60) must be satisfied. According to these two principles, we can obtain

$$\left\{Z_m^{(i)}(\xi)\right\} = \left[S^{(i)}(k_m^i, \xi)\right]\left\{Z_m^{(i)}(\xi_{i-1})\right\}, \tag{63}$$

where $\{Z_m^{(i)}(\xi_{i-1})\}$ is the so-called initial parameter, and

$$\left\{Z_m^{(i)}(\xi)\right\} = \begin{Bmatrix} R_m^{(i)}(\xi) \\ \sigma_m^{(i)}(\xi) \end{Bmatrix}, \qquad \left[S^{(i)}(k_m^i, \xi)\right] = \begin{bmatrix} S_{11}^{(i)}(k_m^i, \xi) & S_{12}^{(i)}(k_m^i, \xi) \\ S_{21}^{(i)}(k_m^i, \xi) & S_{22}^{(i)}(k_m^i, \xi) \end{bmatrix}, \tag{64}$$

in which

$$\begin{aligned}
S_{11}^{(i)}(k_m^i, \xi) &= \left[P_Y(i, k_m^i, \xi_{i-1})J_{\mu_i}(k_m^i\xi) - P_J(i, k_m^i, \xi_{i-1})Y_{\mu_i}(k_m^i\xi)\right]/\Lambda_i, \\
S_{12}^{(i)}(k_m^i, \xi) &= \left[J_{\mu_i}(k_m^i\xi_{i-1})Y_{\mu_i}(k_m^i\xi) - Y_{\mu_i}(k_m^i\xi_{i-1})J_{\mu_i}(k_m^i\xi)\right]/\Lambda_i, \\
S_{21}^{(i)}(k_m^i, \xi) &= \left[P_Y(i, k_m^i, \xi_{i-1})P_J(i, k_m^i, \xi) - P_J(i, k_m^i, \xi_{i-1})P_Y(i, k_m^i, \xi)\right]/\Lambda_i, \\
S_{22}^{(i)}(k_m^i, \xi) &= \left[J_{\mu_i}(k_m^i\xi_{i-1})P_Y(i, k_m^i, \xi) - Y_{\mu_i}(k_m^i\xi_{i-1})P_J(i, k_m^i, \xi)\right]/\Lambda_i,
\end{aligned} \tag{65}$$

and

$$\begin{aligned}&\Lambda_i = P_Y(i,k_m^i,\xi_{i-1})J_{\mu_i}(k_m^i\xi_{i-1}) - P_J(i,k_m^i,\xi_{i-1})Y_{\mu_i}(k_m^i\xi_{i-1}),\\&P_J(i,k_m^i,\xi) = c_{33D}^{(i)}\nabla J_{\mu_i}(k_m^i\xi) + c_{13D}^{(i)}J_{\mu_i}(k_m^i\xi),\\&P_Y(i,k_m^i,\xi) = c_{33D}^{(i)}\nabla Y_{\mu_i}(k_m^i\xi) + c_{13D}^{(i)}Y_{\mu_i}(k_m^i\xi).\end{aligned} \tag{66}$$

In view of the first equation in Eq. (64), Eq. (62) can be rewritten as

$$\{Z_m^{(i+1)}(\xi_i)\} = \{Z_m^{(i)}(\xi_i)\} \quad (i = 1,2,\ldots,n-1). \tag{67}$$

Setting $\xi = \xi_i$ in Eq. (63) and then repeatedly using Eq. (67), we can obtain the following equation:

$$\{Z_m^{(i)}(\xi_i)\} = [Q^{(i)}]\{Z_m^{(1)}(\xi_0)\} \quad (i = 1,2,\ldots,n), \tag{68}$$

where

$$[Q^{(i)}] = \prod_{j=i}^{1}[S^{(j)}(k_m^j,\xi_j)], \tag{69}$$

in which $[Q^{(i)}]$ is a 2×2 matrix. If $i = n$, Eq. (68) then becomes

$$\begin{Bmatrix} R_m^{(n)}(\xi_n) \\ 0 \end{Bmatrix} = \begin{bmatrix} Q_{11}^{(n)} & Q_{12}^{(n)} \\ Q_{21}^{(n)} & Q_{22}^{(n)} \end{bmatrix}\begin{Bmatrix} R_m^{(1)}(\xi_0) \\ 0 \end{Bmatrix}. \tag{70}$$

From the second equation in Eq. (70), we have

$$Q_{21}^{(n)} = 0. \tag{71}$$

Eq. (71), a transcendental equation, is just the eigenequation, from which a series of positive real roots $\omega_m\,(m = 1,2,\ldots,\infty)$ can be obtained. After $\omega_m\,(m = 1,2,\ldots,\infty)$, arranged in an ascending order, have been obtained, Eq. (63) can be rewritten in the following form by virtue of Eq. (68):

$$\begin{Bmatrix} R_m^{(i)}(\xi) \\ \sigma_m^{(i)}(\xi) \end{Bmatrix} = \begin{bmatrix} S_{11}^{(i)}(k_m^i\xi) & S_{12}^{(i)}(k_m^i\xi) \\ S_{21}^{(i)}(k_m^i\xi) & S_{22}^{(i)}(k_m^i\xi) \end{bmatrix}\begin{bmatrix} Q_{11}^{(i-1)} & Q_{12}^{(i-1)} \\ Q_{21}^{(i-1)} & Q_{22}^{(i-1)} \end{bmatrix}\begin{Bmatrix} R_m^{(1)}(\xi_0) \\ 0 \end{Bmatrix}. \tag{72}$$

Then from the first equation in Eq. (72), we derive

$$R_m^{(i)}(\xi) = \left[Q_{11}^{(i-1)}S_{11}^{(i)}(k_m^i\xi) + Q_{21}^{(i-1)}S_{12}^{(i)}(k_m^i\xi)\right]R_m^{(1)}(\xi_0). \tag{73}$$

In Eq. (73), $R_m^{(1)}(\xi_0)$ is a common constant for each layer, which can be taken as $R_m^{(1)}(\xi_0) = 1$ in the calculation. Thus $R_m^{(i)}(\xi)$ is determined completely. Substituting Eq. (57) into Eq. (55) leads to

$$\sum_{m=1}^{\infty} R_m^{(i)}(\xi)\left[\frac{\mathrm{d}^2\Omega_m(\tau)}{\mathrm{d}\tau^2} + \omega_m^2\Omega_m(\tau)\right] = -f_1^{(i)}(\xi)\frac{\mathrm{d}^2p_0(\tau)}{\mathrm{d}\tau^2} - f_2^{(i)}\frac{\mathrm{d}^2p_n(\tau)}{\mathrm{d}\tau^2} - f_3^{(i)}\frac{\mathrm{d}^2\eta(\tau)}{\mathrm{d}\tau^2}. \tag{74}$$

By virtue of the orthogonal properties of Bessel functions, it is easy to verify that $R_m^{(i)}(\xi)$ has the following properties (Yin and Yue, 2002):

$$\sum_{i=1}^{n}\bar{\rho}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\xi R_m^{(i)}(\xi)R_j^{(i)}(\xi)\,\mathrm{d}\xi = J_m\delta_{mj}, \tag{75}$$

where δ_{ml} is the Kronecker delta, and

$$J_m = \frac{1}{2}\sum_{i=1}^{n}\left\{\frac{c_{33D}^{(i)}}{\omega_m^2}\left[\xi\frac{\mathrm{d}}{\mathrm{d}\xi}R_m^{(i)}(\xi)\right]^2 - \frac{\mu_i^2c_{33D}^{(i)}}{\omega_m^2}\left[R_m^{(i)}(\xi)\right]^2 + \bar{\rho}^{(i)}\left[\xi R_m^{(i)}(\xi)\right]^2\right\}\Bigg|_{\xi_{i-1}}^{\xi_i}. \tag{76}$$

Utilizing Eq. (75), we can derive the following equation from Eq. (74):

$$\frac{\mathrm{d}^2\Omega_m(\tau)}{\mathrm{d}\tau^2}+\omega_m^2\Omega_m(\tau)=q_m(\tau)\quad (m=1,2,\ldots,\infty), \tag{77}$$

where

$$\begin{aligned}
&q_m(\tau)=q_{1m}(\tau)+I_{3m}\ddot{\eta}(\tau),\qquad q_{1m}(\tau)=I_{1m}\ddot{p}_0(\tau)+I_{2m}\ddot{p}_n(\tau)\\
&I_{1m}=-\sum_{i=1}^{n}\bar{\rho}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\xi f_1^{(i)}(\xi)R_m^{(i)}(\xi)\mathrm{d}\xi\bigg/J_m,\\
&I_{2m}=-\sum_{i=1}^{n}\bar{\rho}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\xi f_2^{(i)}(\xi)R_m^{(i)}(\xi)\mathrm{d}\xi\bigg/J_m,\\
&I_{3m}=-\sum_{i=1}^{n}\bar{\rho}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\xi f_3^{(i)}(\xi)R_m^{(i)}(\xi)\mathrm{d}\xi\bigg/J_m.
\end{aligned} \tag{78}$$

The solution of Eq. (77) is

$$\Omega_m(\tau)=\Omega_m(0)\cos\omega_m\tau+\frac{\dot{\Omega}_m(0)}{\omega_m}\sin\omega_m\tau+\frac{1}{\omega_m}\int_0^\tau q_m(p)\sin\omega_m(\tau-p)\,\mathrm{d}p. \tag{79}$$

Utilizing Eqs. (49), (57) and (34), we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}R_m^{(i)}(\xi)\Omega_m(0)=u_0^{(i)}(\xi)-f_1^{(i)}(\xi)p_0(0)-f_2^{(i)}(\xi)p_n(0)-f_3^{(i)}(\xi)\eta(0),\\
&\sum_{m=1}^{\infty}R_m^{(i)}(\xi)\dot{\Omega}_m(0)=v_0^{(i)}(\xi)-f_1^{(i)}(\xi)\dot{p}_0(0)-f_2^{(i)}(\xi)\dot{p}_n(0)-f_3^{(i)}(xi)\dot{\eta}(0).
\end{aligned} \tag{80}$$

By using Eq. (75), we can obtain $\Omega_m(0)$ and $\dot{\Omega}_m(0)$ from Eq. (80),

$$\begin{aligned}
&\Omega_m(0)=I_{1m}p_0(0)+I_{2m}p_n(0)+I_{3m}\eta(0)+I_{4m},\\
&\dot{\Omega}_m(0)=I_{1m}\dot{p}_0(0)+I_{2m}\dot{p}_n(0)+I_{3m}\dot{\eta}(0)+I_{5m},
\end{aligned} \tag{81}$$

where

$$I_{4m}=\sum_{i=1}^{n}\bar{\rho}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\xi u_0^{(i)}(\xi)R_m^{(i)}(\xi)\,\mathrm{d}\xi\bigg/J_m,\quad I_{5m}=\sum_{i=1}^{n}\bar{\rho}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\xi v_0^{(i)}(\xi)R_m^{(i)}(\xi)\,\mathrm{d}\xi\bigg/J_m. \tag{82}$$

Notice that $\ddot{\eta}(p)$ is involved in $q_m(p)$ as shown in the first equation in Eq. (78). We apply the integration-by-parts formula to perform the integration of the term involving $\ddot{\eta}(p)$ in Eq. (79) and derive

$$\begin{aligned}
&\int_0^\tau\ddot{\eta}(p)\sin\omega_m(\tau-p)\,\mathrm{d}p\\
&\quad=-\dot{\eta}(0)\sin\omega_m\tau-\eta(0)\omega_m\cos\omega_m\tau+\omega_m\eta(\tau)-\omega_m^2\int_0^\tau\eta(p)\sin\omega_m(\tau-p)\,\mathrm{d}p.
\end{aligned} \tag{83}$$

Then the following equation can be obtained from Eq. (79),

$$\Omega_m(\tau)=\Omega_{1m}(\tau)+I_{3m}\eta(\tau)-I_{3m}\omega_m\int_0^\tau\eta(p)\sin\omega_m(\tau-p)\,\mathrm{d}p, \tag{84}$$

where

$$\Omega_{1m}(\tau) = \Omega_m(0)\cos\omega_m\tau + \frac{\dot{\Omega}_m(0)}{\omega_m}\sin\omega_m\tau + \frac{1}{\omega_m}\int_0^\tau q_{1m}(p)\sin\omega_m(\tau - p)\,\mathrm{d}p$$
$$- \frac{I_{3m}}{\omega_m}[\dot{\eta}(0)\sin\omega_m\tau + \eta(0)\omega_m\cos\omega_m\tau]. \tag{85}$$

Now we will determine $\eta(0)$, $\dot{\eta}(0)$ and $\eta(\tau)$ by means of the electric boundary condition (12b) and the electric potential continuity condition (13b). We first rewrite Eq. (22) as

$$\frac{\partial\phi^{(i)}}{\partial\xi} = \frac{e_1^{(i)}}{\varepsilon_3^{(i)}}\frac{u^{(i)}}{\xi} + \frac{e_3^{(i)}}{\varepsilon_3^{(i)}}\frac{\partial u^{(i)}}{\partial\xi} - \frac{1}{\varepsilon_3^{(i)}}\frac{\eta(\tau)}{\xi}, \tag{86}$$

in which $u^{(i)}$ can be obtained by virtue of Eqs. (25), (49) and (57) as follows:

$$u^{(i)}(\xi,\tau) = \sum_{m=1}^{\infty} R_m^{(i)}(\xi)\Omega_m(\tau) + f_1^{(i)}(\xi)p_0(\tau) + f_2^{(i)}(\xi)p_n(\tau) + f_3^{(i)}(\xi)\eta(\tau). \tag{87}$$

In view of Eqs. (12b) and (13b), we integrate Eq. (87) at the interval $[\xi_{i-1}, \xi_i]$ $(i = 1, 2, \ldots, n)$ and then summarize them. The following equation is then derived:

$$\psi_1(\tau) = K_1\eta(\tau) + \sum_{m=1}^{\infty} K_{2m}\Omega_m(\tau), \tag{88}$$

where

$$\begin{aligned}
&\psi_1(\tau) = \phi_n(\tau) - \phi_0(\tau) - K_3p_0(\tau) - K_4p_n(\tau),\\
&K_1 = \sum_{i=1}^{n}\left\{\frac{e_1^{(i)}}{\varepsilon_3^{(i)}}\int_{\xi_{i-1}}^{\xi_i}\frac{f_3^{(i)}(\xi)}{\xi}\mathrm{d}\xi + \frac{e_3^{(i)}}{\varepsilon_3^{(i)}}\left[f_3^{(i)}(\xi_i) - f_3^{(i)}(\xi_{i-1})\right] - \frac{1}{\varepsilon_3^{(i)}}\ln\left(\frac{\xi_i}{\xi_{i-1}}\right)\right\},\\
&K_{2m} = \sum_{i=1}^{n}\left\{\frac{e_1^{(i)}}{\varepsilon_3^{(i)}}\int_{\xi_{i-1}}^{\xi_i}\frac{R_m^{(i)}(\xi)}{\xi}\mathrm{d}\xi + \frac{e_3^{(i)}}{\varepsilon_3^{(i)}}\left[R_m^{(i)}(\xi_i) - R_m^{(i)}(\xi_{i-1})\right]\right\},\\
&K_3 = \sum_{i=1}^{n}\left\{\frac{e_1^{(i)}}{\varepsilon_3^{(i)}}\int_{\xi_{i-1}}^{\xi_i}\frac{f_1^{(i)}(\xi)}{\xi}\mathrm{d}\xi + \frac{e_3^{(i)}}{\varepsilon_3^{(i)}}\left[f_1^{(i)}(\xi_i) - f_1^{(i)}(\xi_{i-1})\right]\right\},\\
&K_4 = \sum_{i=1}^{n}\left\{\frac{e_1^{(i)}}{\varepsilon_3^{(i)}}\int_{\xi_{i-1}}^{\xi_i}\frac{f_2^{(i)}(\xi)}{\xi}\mathrm{d}\xi + \frac{e_3^{(i)}}{\varepsilon_3^{(i)}}\left[f_2^{(i)}(\xi_i) - f_2^{(i)}(\xi_{i-1})\right]\right\}.
\end{aligned} \tag{89}$$

From Eq. (88), we have

$$\dot{\psi}_1(\tau) = K_1\dot{\eta}(\tau) + \sum_{m=1}^{\infty} K_{2m}\dot{\Omega}_m(\tau). \tag{90}$$

If $\tau = 0$, by means of Eq. (81), we can determine $\eta(0)$ and $\dot{\eta}(0)$ from Eqs. (88) and (90),

$$\begin{aligned}
&\eta(0) = \frac{\psi_1(0) - \sum_{m=1}^{\infty}K_{2m}[I_{1m}p_0(0) + I_{2m}p_n(0) + I_{4m}]}{K_1 + \sum_{m=1}^{\infty}K_{2m}I_{3m}},\\
&\dot{\eta}(0) = \frac{\dot{\psi}_1(0) - \sum_{m=1}^{\infty}K_{2m}[I_{1m}\dot{p}_0(0) + I_{2m}\dot{p}_n(0) + I_{5m}]}{K_1 + \sum_{m=1}^{\infty}K_{2m}I_{3m}}.
\end{aligned} \tag{91}$$

Substituting Eq. (91) into Eqs. (81) and (85), the constants $\Omega_m(0)$ and $\dot{\Omega}_m(0)$ and the function $\Omega_{1m}(\tau)$ become known. Then substituting Eq. (84) into Eq. (88) leads to

$$\psi(\tau) = N_1\eta(\tau) + \sum_{m=1}^{\infty} N_{2m} \int_0^{\tau} \eta(p)\sin\omega_m(\tau - p)\,\mathrm{d}p, \tag{92}$$

where

$$\begin{aligned} \psi(\tau) &= \psi_1(\tau) - \sum_{m=1}^{\infty} K_{2m}\Omega_{1m}(\tau), \\ N_1 &= K_1 + \sum_{m=1}^{\infty} K_{2m}I_{3m}, \qquad N_{2m} = -\omega_m K_{2m} I_{3m}. \end{aligned} \tag{93}$$

It is noted that Eq. (92) is a Volterra integral equation of the second kind (Kress, 1989), of which analytical solutions can be obtained only for some special cases. Generally, numerical methods should often be adopted. Recently, we have constructed a recursive formula by which Eq. (92) can be solved efficiently and quickly (Ding et al., 2003b). After $\eta(\tau)$ is obtained, the displacement, electric potential and stresses can then be completely determined.

6. Numerical results and analysis

Example 1. In this example, we consider a homogeneous piezoelectric infinite hollow cylinder, of which an analytical solution has been obtained by Ding et al. (2003a). The material is taken as PZT-4 (Table 1). In the demonstration, we divide the cylinder into five layers ($n = 5$) when using the present method. The non-dimensional inner radius, the radii of the interfaces as well as the outer radius are $\xi_0 = 0.5$, $\xi_1 = 0.625$, $\xi_2 = 0.75$, $\xi_3 = 0.875$, $\xi_4 = 1.0$, respectively. The hollow cylinder, which is at rest at $t = 0$, i.e. $u_0^{(i)}(\xi) = 0$, $v_0^{(i)}(\xi) = 0$ ($i = 1,2,\ldots,n$), is subjected to a sudden constant pressure at the inner surface. The boundary conditions are

$$p_0(\tau) = H(\tau), \qquad p_5(\tau) = 0, \tag{94a}$$

$$\phi^{(1)}(\xi_0,\tau) = 0, \qquad \phi^{(5)}(\xi_5,\tau) = 0, \tag{94b}$$

where $H(\cdot)$ denotes the Heaviside function.

Table 1
Elastic, piezoelectric and dielectric constants of piezoelectric materials

Material constant	Layer 1	Layer 2	Layer 3	Layer 4	Layer 5
	PZT-4	$BaTiO_3$	PZT-5H	$BaTiO_3$	PZT-4
c_{11} (GPa)	139.0	150.0	126.0	150.0	139.0
c_{12} (GPa)	77.8	66.0	79.5	66.0	77.8
c_{13} (GPa)	74.3	66.0	84.1	66.0	74.3
c_{23} (GPa)	74.3	66.0	84.1	66.0	74.3
c_{33} (GPa)	115.0	146.0	117.0	146.0	115.0
e_{31} (C/m^2)	−5.2	−4.35	−6.5	−4.35	−5.2
e_{32} (C/m^2)	−5.2	−4.35	−6.5	−4.35	−5.2
e_{33} (C/m^2)	15.1	17.5	23.3	17.5	15.1
ε_{33} ($\times 10^{-9}$ F/m)	5.62	15.04	13.0	15.04	5.62
ρ ($\times 10^3$ kg/m^3)	7.5	5.7	7.5	5.7	7.5

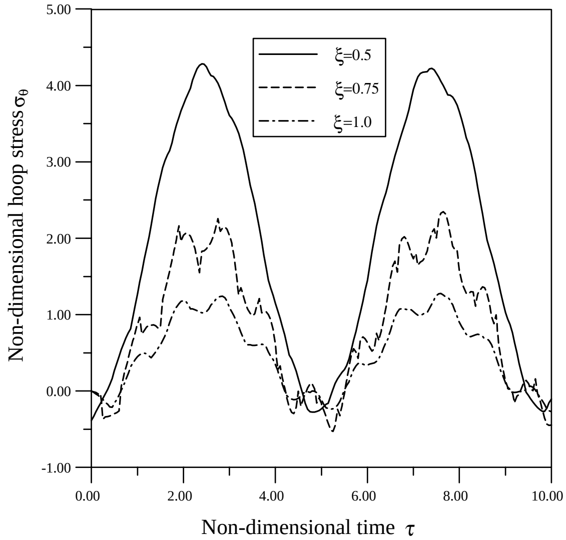

Fig. 2. Time histories of hoop stress σ_θ at different locations in the homogeneous piezoelectric hollow cylinder.

The time histories of the non-dimensional hoop stresses at the inner surface ($\xi = 0.5$), middle surface ($\xi = 0.75$), as well as outer surface ($\xi = 1.0$) are shown in Fig. 2. The results agree well with those presented in Ding et al. (2003a). Thus, the validity of the present solution is clarified.

Example 2. Now we consider the dynamic responses of a five-layer infinite hollow cylinder ($n = 5$) with $\xi_0 = 0.5$, $\xi_1 = 0.6$, $\xi_2 = 0.7$, $\xi_3 = 0.8$, $\xi_4 = 0.9$ and $\xi_5 = 1.0$, respectively, subjected to a sudden constant pressure at the inner surface. The material constants of each layer are listed in Table 1 (Adelman et al., 1975a; Kharouf and Heyliger, 1994). The boundary conditions as well as other parameters all are the same as those in Example 1.

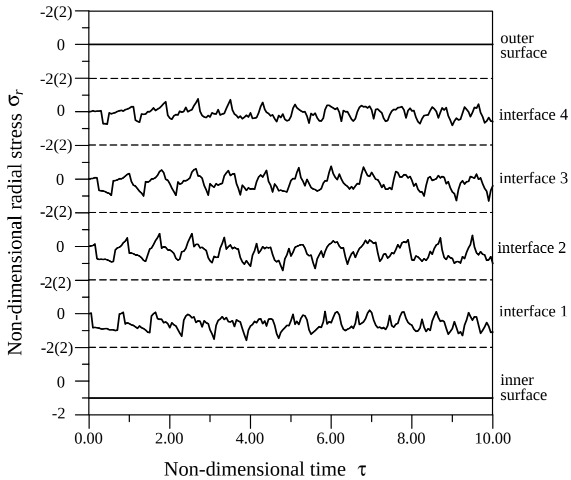

Fig. 3. Time histories of radial stress σ_r at the each surface of the five-layered piezoelectric hollow cylinder.

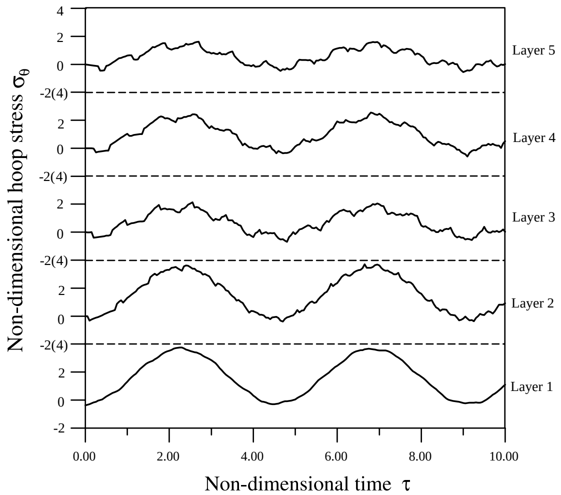

Fig. 4. Time histories of hoop stress σ_θ at the inner surface of each layer of the five-layered piezoelectric hollow cylinder.

The time histories of the radial stress σ_r at the inner and outer surfaces as well as at each interface are shown in Fig. 3. From the curves, we find that the non-dimensional radial stress at the inner surface keeps -1 while that at the outer surface keeps zero. Thus, the calculated results satisfy the prescribed mechanical boundary conditions, thus the correctness of the numerical results is clarified in this respect.

Figs. 4 and 5 depict the time histories of the hoop stress σ_θ at the inner and outer surfaces of each layer, respectively. It can be seen that in the same layer, the amplitude of σ_θ at the inner surface is always larger than that at the outer surface.

At the initial phase, the distributions of non-dimensional radial stresses σ_r at $\tau = 0.05$, 0.15, 0.25, 0.35, 0.4, 0.5, 0.6 and 0.7 in the five-layer piezoelectric hollow cylinder are shown in Fig. 6a and b. It is clearly seen that the stress wave generates just when a constant pressure suddenly acts on the inner surface and then propagates

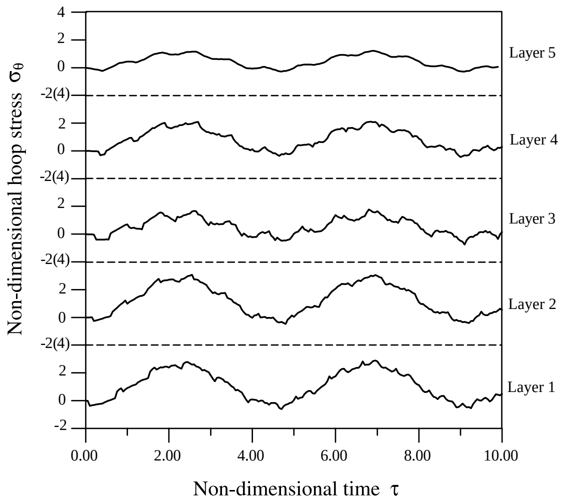

Fig. 5. Time histories of hoop stress σ_θ at the outer surface of each layer of the five-layered piezoelectric hollow cylinder.

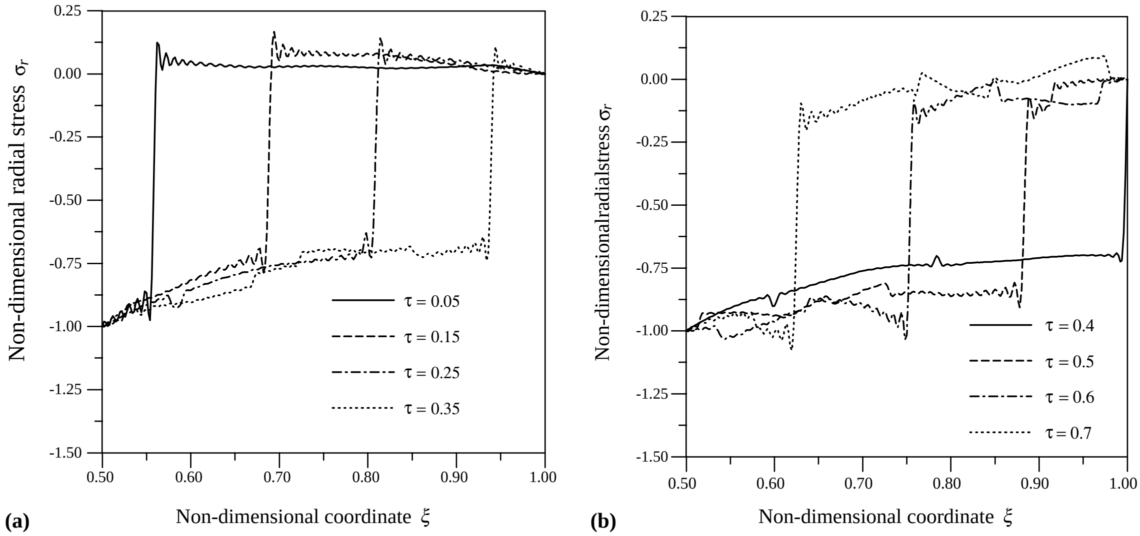

Fig. 6. Distributions of non-dimensional radial stresses σ_r at different times in the five-layer piezoelectric hollow cylinder.

from the inner to the outer. When they arrive at the outer surface, it is reflected backward and propagates along the opposite direction. We also find that when the stress wavefront arrives, it causes a strong discontinuity in the stress. Fig. 7a and b depict the distributions of non-dimensional hoop stresses σ_θ at $\tau = 0.05$, 0.15, 0.25, 0.35, 2.0, 4.0 and 8.0, respectively. In Fig. 7a and b, the discontinuity of the hoop stress also appears when the stress wavefront arrives. We also notice that the hoop stress always jumps at each interface.

The distributions of the non-dimensional electric potential ϕ at different times ($\tau = 0.25$, 0.5 and 1.0) are shown in Fig. 8. From the curves, we find that the calculated electric potentials at the inner and outer

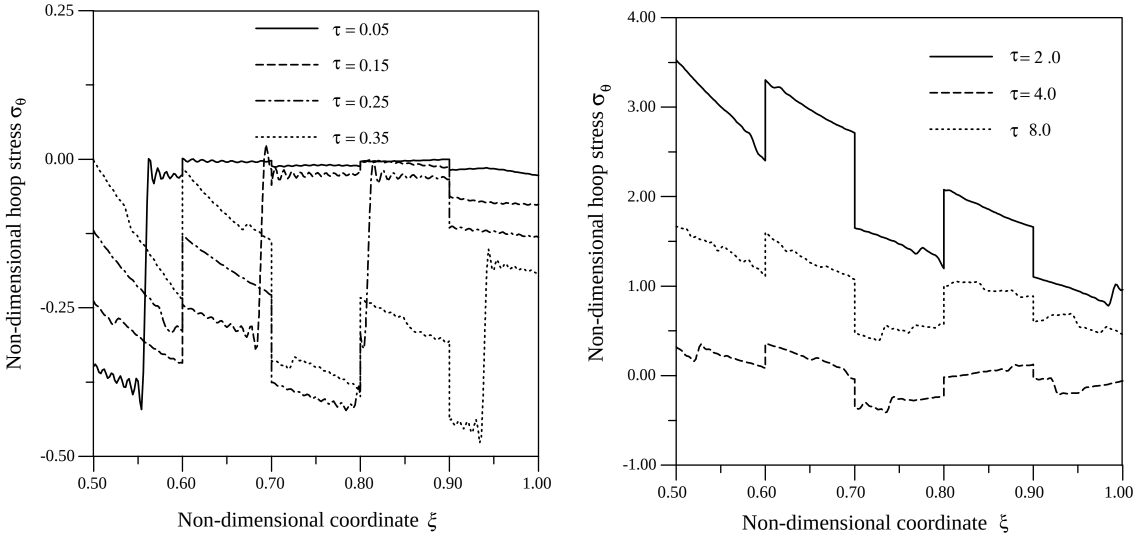

Fig. 7. Distributions of non-dimensional hoop stresses σ_θ at different times in the five-layer piezoelectric hollow cylinder.

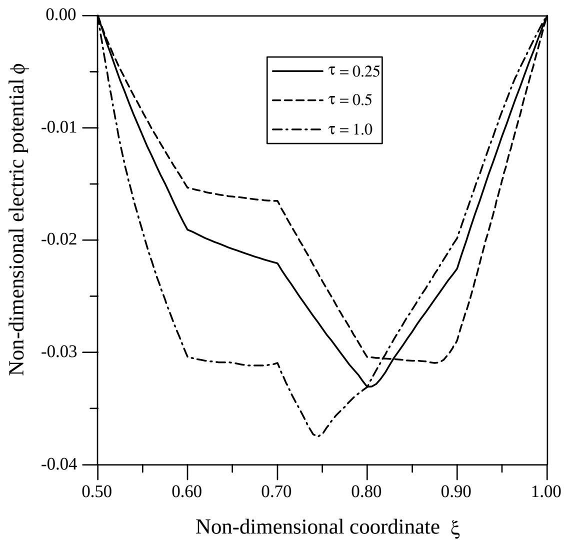

Fig. 8. Distributions of non-dimensional electric potential ϕ at different times in the five-layered piezoelectric hollow cylinder.

surfaces both are zero, which are the right electric boundary conditions. The correctness of the numerical results is further clarified.

7. Conclusions

In this paper, the principle of superposition is successfully applied to analyze the axisymmetric plane strain dynamic problems for multilayered orthotropic piezoelectric infinite hollow cylinder. The quasi-static solution is obtained in an explicit form by using the state space method. The derivation procedure is completed via operating the matrix of order two only in spite of the layer number. While in deriving the dynamic solution, the initial parameter method is introduced to deal with the continuity condition at the interfaces, and the eigenequation is obtained in a very concise form also via operating the matrix of order two only, from which the eigenvalues can be obtained quickly with a high accuracy.

From Figs. 3–5, we can clearly see the oscillations of radial and hoop stresses in a five-layer orthotropic piezoelectric infinite hollow cylinder. The phenomena can be well explained by the wave motion viewpoint: when a constant pressure suddenly applied onto the inner surface, the stress waves are then generated and propagate from the inner to the outer. When they arrive at the outer surface (the interfaces), the reflected (and transmitted) waves are further generated. It is the multiple reflection and transmission of the stress waves that leads to the stresses oscillating in the multilayered hollow cylinder. It should be mentioned here that the numerical results for a five-layer hollow cylinder in Example 2 have been obtained very quickly in our calculation. So, the present method provides an effective way for analyzing the transient responses of the laminated infinite piezoelectric hollow cylinders.

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